Worksheet for Sections 7.3 and 7.5

1. (a) Let $f(t) = a(b^t)$, with $f(-1) = 1/2$ and $f(2) = 108$. Find the values of a and b.
   (b) Let $x > 0$. Order the following functions from smallest to largest, and sketch their graphs:

$$3^{-x}, \qquad 3^x, \qquad 2^{-x}, \qquad 2^x, \qquad e^x$$

2. (a) Let a and b be positive constants, with $a \neq 1$ and $b \neq 1$. Using Theorem 7.8, prove the general change of base formula

$$\log_b x = \log_b a \log_a x, \quad \text{for all}\, x > 0$$

   (b) We know that $\log_2 7 \approx 2.807355$, $\log_{15} 7 \approx 0.718565$, and $\log_7 15 \approx 1.391663$. Using (a) and whichever such approximations are relevant, approximate $\log_2 15$.

3. For certain numerical values of c, the answer when we evaluate the integral $\displaystyle\int \frac{1}{x^2 + 6x + c}\, dx$ involves an arctangent. By completing the square in the denominator of the integrand, determine those values of c, and evaluate the corresponding integral for an arbitrary such value of c.

4. Let g be a differentiable function defined on $[0, 1]$, with $|g(t)| < 3$ for $0 \leq t \leq 1$. Thus $16 - (g(t))^2$ is strictly positive on $[0, 1]$.

   (a) Substitute $u = g(t)$, and then evaluate the integral $\displaystyle\int \frac{g'(t)}{\sqrt{16 - (g(t))^2}}\, dt$.

   (b) Suppose $g(0) = 0$. Find a value of $g(1)$ so that $\displaystyle\int_0^1 \frac{g'(t)}{\sqrt{16 - (g(t))^2}}\, dt = \frac{\pi}{3}$.

5. Jo drops a marble from her apartment window, 20 meters above the ground. At the same height but 12 meters away, Doe watches the marble fall. When the height of the marble above the ground is h, let θ be the angle between L and M, where L is the horizontal line joining Jo and Doe, and where M represents Doe's line of sight to the marble.

   (a) Draw a picture of the scene, including L, M, and θ. Put the horizontal axis at ground level, and indicate the position of the marble $h(t)$ feet above ground level at time t.
   (b) Find θ in terms of h, and then calculate $d\theta/dh$.
   (c) Assume that $h(t) = 20 - 4.9t^2$ until the marble hits the ground. Find $d\theta/dt$. Show geometrically why $d\theta/dt > 0$, and then tell why the formula for $d\theta/dt$ tells us that $d\theta/dt > 0$.

   Note: It turns out that there is t_0 for which $d\theta/dt$ has a maximum value. For $\theta(t_0)$ the marble appears to be falling the fastest. Can you find the value of t_0?